Low-Frequency Characteristic Modes for Aperture Coupling Problems

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Abstract —**In this paper, a recently suggested general procedure which leads to an eigenvalue equation for aperture problems is specialized to the range of frequencies for which the maximum linear dimension of the aperture l is much smaller than the wavelengths, known as the Rayleigh region. As kl asymptotically approaches zero, we arrive at a set of two eigenvalue equations which, aided by the edge condition, constitutes an alternative set of equations for a derivation of the quasi-static distributions characterizing the aperture.**

I. Introduction

Modal solutions have long been used for the analysis and synthesis of radiating systems. The most familiar case is when the regions of the source and the field coincide with coordinate surfaces in which the Helmholtz equation is separable. For bodies of arbitrary shape, similar modes can be defined. In these cases, the modal solutions are eigenvectors of a generalized or weighted eigenvalue equation. Garbacz [1] approached the problem by diagonalizing the scattering matrix of the body, and his results for wire objects are given in [2]. Harrington and Mautz [3] dealt with the problem by diagonalizing the generalized impedance matrix of the body, and their results for wire objects as well as bodies of revolution are given in [4]. Harrington *et al.* [5] subsequently extended the formulation to encompass dielectric, magnetic, and both dielectric and magnetic bodies. Other related work includes that of Inagaki and Garbacz [6] and Eftimiu and Huddleston [7]. Inagaki and Garbacz extended Parseval's relation and arrived at an eigenvalue equation for which the eigensources and corresponding eigenfields are complete and orthogonal over the source and field regions, respectively. Then they applied the theory to arrays and to a two-dimensional aperture problem. Eftimiu and Huddleston developed a useful approximate analytic expression for both eigenvalues and eigenvectors for the case of a long finite circular cylinder. Recently, Harrington and Mautz [8] suggested a procedure similar to that of [3] which leads to an eigenvalue equation for equivalent magnetic currents in three-dimensional aperture problems.

The coupling of electromagnetic energy through an aperture in a conducting wall is an important problem in the theory of electromagnetic compatibility and interference. A model used in recent years is that of two regions separated by an infinitely thin, perfectly conducting wall in which an aperture is cut. The method of solution is briefly as follows. The equivalence principle is used to divide the original problem into two parts; this is done by replacing the aperture by a perfect conductor and providing for the tangential electric field originally present in the aperture by attaching postulated magnetic current sheets to both sides of the aperture. Continuity of the tangential magnetic field across the aperture gives an integral equation for the unknown magnetic current. To solve the integral equation via the method of moments, the unknown magnetic current is expressed as a linear combination of a selected set of expansion vector functions. This linear combination is then substituted into the integral equation, which in turn is tested with each element of a set of testing functions. Obviously, the success and simplicity of the moment solution depend, often crucially, on a suitable choice of both expansion and testing functions. In this context, the eigenfunctions yielded by the eigenvalue equation possess the following desirable properties. They are equiphasal and can be chosen real; they are orthogonal in some sense over the aperture region; and their radiation fields are Hermitian orthogonal over the sphere at infinity. Hence, using these functions can unquestionably render moment solutions for aperture problems very simple. This assured simplification would come of course at the expense of the indirect step of determining the eigencurrents. However, it is expected that for electrically small apertures, only a few modes, which can be ordered according to their relative importance, are required for accurate solutions.

In this paper, we specialize the procedure of [8] to the Rayleigh region, i.e., the range of frequencies for which the maximum linear dimension of the aperture is much smaller than the wavelength. The eigenvalue equation is first ordered in ascending powers of kl, where k is the wavenumber and l is the maximum linear dimension of the aperture. Then, we allow kl to approach zero, thereby obtaining two low-frequency eigenvalue equations which, aided by the edge condition, constitute an alternative set of equations for the derivation of the quasi-static distributions characterizing the aperture. Specifically, the two eigencurrents of the first low-frequency eigenvalue equation give rise, by means of the equation of continuity, to the two quasi-static magnetic charge densities, while the solenoidal eigencurrent of the second eigenvalue equation is the actual quasi-static magnetic current.

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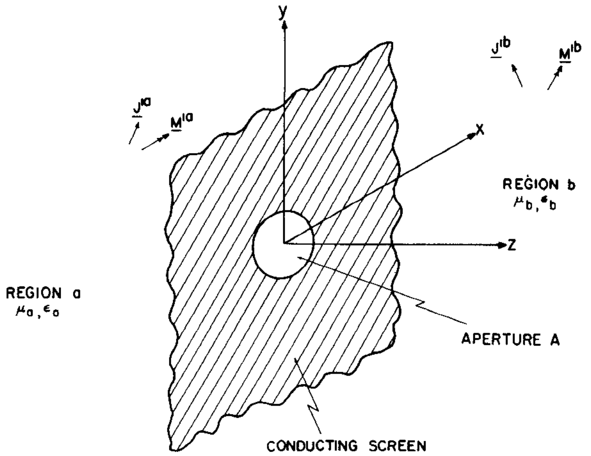

Fig. 1. Two half-space regions coupled through an electrically small aperture.

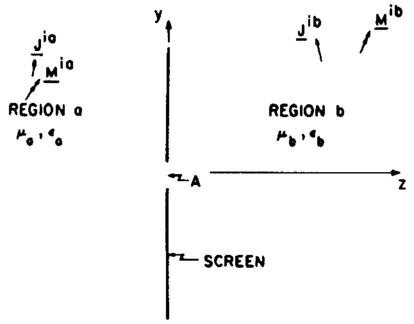

Fig. 2. Original situation.

II. Characteristic Modes for Aperture Region

A model for aperture coupling problems is that of two regions, called region a and region b, separated by an infinitely thin, perfectly conducting wall in which an aperture A is cut. The geometry is shown in Fig. 1. A corresponding cross section is shown in Fig. 2. Regions a and b are filled with contrasting homogeneous media with constitutive parameters (μ_a, ϵ_a) and (μ_b, ϵ_b). The excitation is due to known sources $(\boldsymbol{J}^{ia}, \boldsymbol{M}^{ia})$ and $(\boldsymbol{J}^{ib}, \boldsymbol{M}^{ib})$ with $\exp(j\omega t)$ time dependence in regions a and b, respectively. The method of solution is briefly as follows. The equivalence principle is used to divide the situation in the original problem into two equivalent ones, as shown in Figs. 3 and 4. We close the aperture with a perfect conductor and provide for the tangential electric field originally present in the aperture region by attaching postulated magnetic current sheets $-\boldsymbol{M}$ just to the left of the aperture and $\boldsymbol{M}$ just to the right of the aperture, both radiating with the aperture closed. Here

$$\boldsymbol{M} = \boldsymbol{E} \times \hat{\boldsymbol{n}} \tag{1}$$

where $\hat{\boldsymbol{n}}$ is a unit vector normal to A and pointing toward region b, and $\boldsymbol{E}$ is the electric field in the aperture in the original problem. The use of $-\boldsymbol{M}$ in region a and $\boldsymbol{M}$ in region b ensures continuity of the tangential component of the electric field across the aperture. Continuity of the tangential component of the magnetic field $\boldsymbol{H}$ across the aperture leads to the operator equations for the problem.

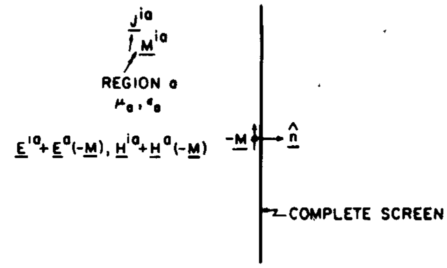

Fig. 3. Equivalence for region a.

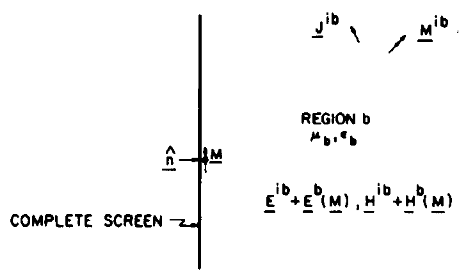

Fig. 4. Equivalence for region b.

The result is

$$-\boldsymbol{H}_t^a(\boldsymbol{M}) - \boldsymbol{H}_t^b(\boldsymbol{M}) = \boldsymbol{H}_t^{ib} - \boldsymbol{H}_t^{ia}, \quad \text{in } A. \tag{2}$$

Here, $\boldsymbol{H}^p(\boldsymbol{M})$ and $\boldsymbol{H}^{ip}$ denote, respectively, the magnetic field due to $\boldsymbol{M}$ and due to the impressed sources radiating in region $p, p = a, b$, with the aperture shorted. The subscript t denotes components tangential to A. Equation (2) is first solved for the equivalent magnetic current $\boldsymbol{M}$ and then the fields in each region can be readily computed.

An operator for the equivalent magnetic current can thus be defined in accordance with (2) as

$$\boldsymbol{Y}(\boldsymbol{M}) = -\boldsymbol{H}_t^a(\boldsymbol{M}) - \boldsymbol{H}_t^b(\boldsymbol{M}), \quad \text{in } A. \tag{3}$$

Further, we define the inner product of two vector functions $\boldsymbol{B}$ and $\boldsymbol{C}$ on A to be the integral of their dot product over the aperture region, i.e.,

$$\langle \boldsymbol{B}, \boldsymbol{C} \rangle = \iint_A \boldsymbol{B} \cdot \boldsymbol{C} \, ds'. \tag{4}$$

Due to reciprocity, the operator $\boldsymbol{H}_t^p(\boldsymbol{M})$, $p = a, b$, is symmetric, i.e.,

$$\langle \boldsymbol{H}_t^p(\boldsymbol{M}_1), \boldsymbol{M}_2 \rangle = \langle \boldsymbol{M}_1, \boldsymbol{H}_t^p(\boldsymbol{M}_2) \rangle \tag{5}$$

and it follows from linearity that $\boldsymbol{Y}(\boldsymbol{M})$ is a symmetric operator as well. On the other hand, $\boldsymbol{H}_t^p(\boldsymbol{M})$, $p = a, b$, is not a Hermitian operator since in general

$$\langle \boldsymbol{H}_t^p(\boldsymbol{M}_1), \boldsymbol{M}_2 \rangle \neq \langle \boldsymbol{M}_1, \boldsymbol{H}_t^{p*}(\boldsymbol{M}_2) \rangle \tag{6}$$

where the asterisk denotes complex conjugate. Consequently, $\boldsymbol{Y}$ is not Hermitian. With a view toward obtaining Hermitian operators, we split $\boldsymbol{Y}$ into its real and imaginary parts as follows:

$$\boldsymbol{Y} = \boldsymbol{G} + j\boldsymbol{B} \tag{7}$$

where

$$\boldsymbol{G} = \tfrac{1}{2}(\boldsymbol{Y} + \boldsymbol{Y}^*) = -\boldsymbol{H}_{t\mathrm{R}}^a - \boldsymbol{H}_{t\mathrm{R}}^b, \quad \text{in } A \tag{8}$$

$$\boldsymbol{B} = \frac{1}{2j}(\boldsymbol{Y} - \boldsymbol{Y}^*) = -\boldsymbol{H}_{t\mathrm{I}}^a - \boldsymbol{H}_{t\mathrm{I}}^b, \quad \text{in } A \tag{9}$$

with H_{tR}^p and H_{tI}^p denoting, respectively, the real and imaginary parts of H_t^p. Evidently, the operators G and B are real and symmetric and thus Hermitian as well. Furthermore, G is positive semidefinite since it is a sum of two positive semidefinite operators, $-H_{tR}^a$ and $-H_{tR}^b$. That each H_{tR}^p is positive semidefinite follows from the fact that the power supplied to region p by the magnetic current M attached to A with the aperture closed, given by

$$P^p = -\mathrm{Re}\langle M^*, H_t^p(M)\rangle = -\langle M^*, H_{tR}^p(M)\rangle \quad (10)$$

is greater than or equal to zero. If no resonator fields exist in either region a or b, i.e., the magnetic currents radiate some power, however small, in either region a or b, then in that region P^p is strictly greater than zero and it follows that G is positive definite.

Next consider the eigenvalue equation

$$Y(M_n) = \nu_n T(M_n) \quad (11)$$

where ν_n are eigenvalues, M_n are eigenfunctions, and T is a weight operator to be chosen. It should be noted that any choice of symmetric T will diagonalize Y. However, for reasons of analytical and conceptual simplicity, $T = G$ is chosen. Further, we set $Y = G + jB$ and $\nu_n = 1 + j\lambda_n$ in (11) and cancel the common term. The result is

$$B(M_n) = \lambda_n G(M_n). \quad (12)$$

Equation (12) constitutes a new weighted eigenvalue equation with λ_n eigenvalues and M_n eigencurrents. In view of the Hermitian property of B and G and the positive definiteness of G, it follows that all eigenvalues must be real and that all eigencurrents are equiphasal over the aperture region and thus can be chosen to be real.

The characteristic currents M_n must also obey the usual orthogonalities and furthermore can be normalized such that

$$\langle M_n^*, G(M_n)\rangle = 1. \quad (13)$$

With this choice, the orthogonality relationships become

$$\langle M_m, G(M_n)\rangle = \langle M_m^*, G(M_n)\rangle = \delta_{mn} \quad (14)$$

$$\langle M_m, B(M_n)\rangle = \langle M_m^*, B(M_n)\rangle = \lambda_n \delta_{mn} \quad (15)$$

$$\langle M_m, Y(M_n)\rangle = \langle M_m^*, Y(M_n)\rangle = (1 + j\lambda_n)\delta_{mn} \quad (16)$$

where δ_{mn} is the Kronecker delta (0 if $m \neq n$ and 1 if $m = n$).

The use of the eigencurrents M_n as both expansion and testing functions in a method of moments solution of (2) will result in a modal solution for M in A as

$$M = \sum_n I_n^i (1 + j\lambda_n)^{-1} M_n \quad (17)$$

where I_n^i are the modal excitation coefficients given by

$$I_n^i = \langle M_n, (H_t^{ib} - H_t^{ia})\rangle. \quad (18)$$

The fields are linearly related to the currents, and hence can also be expressed in modal form, Explicitly, these forms are

$$E^p = \sum_n I_n^i (1 + j\lambda_n)^{-1} E_n^p \quad (19)$$

$$H^p = \sum_n I_n^i (1 + j\lambda_n)^{-1} H_n^p \quad (20)$$

where E^p and H^p are the fields from M and E_n^p and H_n^p are the fields produced by M_n, all radiating in region p in the presence of the complete screen.

III. Specialization to Small Apertures

The magnetic field just to the left and just to the right of the shorted aperture due to the magnetic current M and its associated charge density m $(m = -(1/j\omega)\nabla\cdot M)$ is given by

$$H^p(M) = \frac{k_p}{j\eta_p} \iint_A \frac{M \exp(-jk_p|r - r'|)}{2\pi|r - r'|} ds' + \frac{1}{jk_p\eta_p} \nabla \iint_A \frac{\nabla'\cdot M \exp(-jk_p|r - r'|)}{2\pi|r - r'|} ds',$$

$$p = a, b. \quad (21)$$

Here, r is the position vector of the point in the aperture region at which $H^p(M)$ is evaluated, r' is the vector to the source point in the aperture region, ∇ denotes the gradient operator, $\nabla'\cdot$ denotes the divergence operator in the primed system, ds' is the differential element of area at r', $k_p = \omega\sqrt{\mu_p\epsilon_p}$ is the wavenumber in region p, and $\eta_p = \sqrt{\mu_p/\epsilon_p}$ is the intrinsic impedance of the medium in region p. In the Rayleigh region, i.e., the range of frequencies for which the aperture is electrically small, we expand the $\exp(-jk_p|r - r'|)$ term in (21) in a Taylor series about $k_p|r - r'| = 0$. Using this expansion in (21) and recalling that the eigencurrents are equiphasal over A and can be taken *a priori* to be real, we decompose $H^p(M)$ into its real and imaginary parts as follows:

$$H^p(M) = H_R^p(M) + jH_I^p(M) \quad (22)$$

where $H_R^p(M)$ and $H_I^p(M)$ are given by the series

$$H_R^p(M) = -\frac{(k_p l)^2}{3\pi\eta_p} \iint_A \frac{M}{l^2} ds' + \frac{(k_p l)^4}{60\pi\eta_p} \cdot \iint_A [5M - (r - r')\nabla'\cdot M] \frac{|r - r'|^2}{l^4} ds' + O((k_p l)^6), \quad (k_p l) \to 0, \text{ in } A \quad (23)$$

$$H_I^p(M) = -\frac{1}{2\pi\eta_p(k_p l)} \nabla \iint_A \frac{l\nabla'\cdot M}{|r - r'|} ds' + \frac{(k_p l)}{4\pi\eta_p} \iint_A \frac{[-2M + (r - r')\nabla'\cdot M]}{l|r - r'|} ds' + O((k_p l)^3), \quad (k_p l) \to 0, \text{ in } A. \quad (24)$$

In (23) and (24), l denotes the maximum linear dimension of A. We introduce l to allow the definition of $k_p l$ as a dimensionless small parameter and thereby as a means of ordering the terms according to relative magnitude. Note that the process of obtaining (22)–(24) from (21) involves a few algebraic steps which use several vector identities in conjunction with the fact that on the contour bounding A, the component of $\boldsymbol{M}$ normal to the contour is zero since no line charge can accumulate there. These steps are omitted for the sake of brevity. An interested reader would fill them in with no appreciable difficulty. In the limit $k_p l \to 0$, (23) and (24) are approximated by the first term of their series. We have,

$$H_{\mathrm{R}}^{p}(\boldsymbol{M}) \sim -\frac{k_p^2}{3\pi\eta_p} \iint_A \boldsymbol{M}\, ds', \quad \text{in } A \tag{25}$$

$$H_{\mathrm{I}}^{p}(\boldsymbol{M}) \sim -\frac{1}{2\pi\eta_p k_p} \nabla \iint_A \frac{\nabla' \cdot \boldsymbol{M}}{|\boldsymbol{r}-\boldsymbol{r}'|} ds', \quad \text{in } A \tag{26}$$

provided that $\boldsymbol{M}$ is not a solenoidal function, in which case the right-hand sides of both (25) and (26) are identically zero. If $\boldsymbol{M}$ is a solenoidal function, then in the limit $k_p l \to 0$, (23) and (24) become

$$H_{\mathrm{R}}^{p}(\boldsymbol{M}) \sim \frac{k_p^4}{12\pi\eta_p} \iint_A \boldsymbol{M} |\boldsymbol{r}-\boldsymbol{r}'|^2 ds', \quad \text{in } A \tag{27}$$

$$H_{\mathrm{I}}^{p}(\boldsymbol{M}) \sim -\frac{k_p}{2\pi\eta_p} \iint_A \frac{\boldsymbol{M}}{|\boldsymbol{r}-\boldsymbol{r}'|} ds', \quad \text{in } A. \tag{28}$$

Furthermore, referring to the analysis of the low-frequency portion of the Rayleigh region discussed in [9], we recall that the equivalent magnetic current can be completely described by means of a linear combination of three vector functions tangent to A, denoted by $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, *and* $\boldsymbol{M}_3$. The vector functions $\boldsymbol{M}_1$ *and* $\boldsymbol{M}_2$ can give rise, by means of the equation of continuity ($m = -\nabla \cdot \boldsymbol{M}/j\omega$), to two scalar functions that can span the quasi-static magnetic charge densities which would result in the aperture region under a quasi-static impressed magnetic field. The third function $\boldsymbol{M}_3$ is a divergenceless vector function that on its own can span the quasi-static magnetic current which would result in the aperture region under a quasi-static impressed electric field. Hence, it follows that (25) and (26) are associated with $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, while (27) and (28) are associated with $\boldsymbol{M}_3$.

We can now turn back to the eigenvalue equation (12). Expressing each magnetic field in terms of its real and imaginary parts, given by (25) and (26), we reduce (12) to

$$\frac{3(k_a\eta_a + k_b\eta_b)}{2k_a k_b(\eta_a k_b^2 + \eta_b k_a^2)} \nabla \iint_A \frac{\nabla' \cdot \boldsymbol{M}_n}{|\boldsymbol{r}-\boldsymbol{r}'|} ds'$$

$$= \lambda_n \iint_A (\boldsymbol{r}-\boldsymbol{r}') \nabla' \cdot \boldsymbol{M}_n\, ds', \qquad n = 1,2, \quad \text{in } A. \tag{29}$$

Equation (29) is an eigenvalue equation with, respectively, λ_1 and λ_2 eigenvalues and $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, eigencurrents. As constructed, it constitutes an alternative equation for the derivation of the current distributions $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ which can give rise to two scalar functions that can span the quasi-static magnetic charge densities in the aperture region. Similarly, expressing each magnetic field in terms of its real and imaginary parts, given by (27) and (28), we reduce (12) to

$$-\frac{6(k_a\eta_b + k_b\eta_a)}{k_a^4\eta_b + k_b^4\eta_a} \iint_A \frac{\boldsymbol{M}_3}{|\boldsymbol{r}-\boldsymbol{r}'|} ds'$$

$$= \lambda_3 \iint_A \boldsymbol{M}_3 |\boldsymbol{r}-\boldsymbol{r}'|^2 ds', \quad \text{in } A. \tag{30}$$

Equation (30) is an eigenvalue equation with λ_3 eigenvalue and $\boldsymbol{M}_3$ eigencurrent. As constructed, it constitutes an alternative equation for the derivation of the quasi-static current distribution in the aperture region. Notice that $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, and $\boldsymbol{M}_3$ must obey the edge condition as one approaches the contour bounding the aperture. Specifically, near the edge, the component of each $\boldsymbol{M}_n$ normal to the contour bounding A is of the order $\delta^{1/2}$, where δ is the distance from the edge, while the component of each $\boldsymbol{M}_n$ tangential to the contour bounding A possesses a singularity of the order $\delta^{-1/2}$. In addition, $\boldsymbol{M}_3$ should be a divergenceless vector function.

IV. An Example

As an example, consider a small circular aperture of radius ρ_o in a plane conducting screen. The small circular aperture is suitable for our purposes since its quasi-static aperture distributions are analytically derivable and widely used [10]–[12]. In the following discussion, we will demonstrate that the quasi-static distributions are indeed solutions of the low-frequency eigenvalue equations (29) and (30).

For the circular aperture, due to symmetry, the eigenvalue equation (29) has one eigenvalue whose algebraic multiplicity is 2. The geometric multiplicity of this eigenvalue, i.e., the number of independent eigenvectors for this eigenvalue, is also 2; hence, any linear combination of two independent eigenvectors would also be an eigenvector. Here, we consider the following $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, though any linear combination of them could be adequate as well. These two current distributions give rise, by means of the equation of continuity, to two respective quasi-static charge densities in the aperture region. In terms of cylindrical coordinates, the ρ and φ components of $\boldsymbol{M}_1$ are given by

$$M_{1\rho}(\rho,\varphi) = a_1(\rho_o^2 - \rho^2)^{1/2} \cos\varphi \tag{31}$$

$$M_{1\varphi}(\rho,\varphi) = -a_1\left[(\rho_o^2-\rho^2)^{1/2} + \frac{\rho^2}{2(\rho_o^2-\rho^2)^{1/2}}\right]\sin\varphi \tag{32}$$

and the corresponding components of $\boldsymbol{M}_2$ are given by

$$M_{2\rho}(\rho,\varphi) = a_2(\rho_o^2 - \rho^2)^{1/2} \sin\varphi \tag{33}$$

$$M_{2\varphi}(\rho,\varphi) = a_2\left[(\rho_o^2-\rho^2)^{1/2} + \frac{\rho^2}{2(\rho_o^2-\rho^2)^{1/2}}\right]\cos\varphi. \tag{34}$$

Finally, the ρ and φ components of the solenoidal quasi-static current $\boldsymbol{M}_3$ are given by

$$M_{3\rho} = 0 \tag{35}$$

$$M_{3\varphi} = \frac{\rho}{\left(\rho_o^2 - \rho^2\right)^{1/2}}. \tag{36}$$

The parameters a_1, a_2, and a_3 are real constants taken for normalization purposes to be

$$a_1 = 1/\pi\rho_o^3 \tag{37}$$

$$a_2 = 1/\pi\rho_o^3 \tag{38}$$

$$a_3 = 3/2\pi\rho_o^3. \tag{39}$$

Having specified the functional form of each of the magnetic currents $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, we now return to the eigenvalue equation (29). It can be readily evaluated analytically that

$$\iint_A \boldsymbol{M}_1\, ds' = \hat{\boldsymbol{u}}_x, \quad \text{in } A. \tag{40}$$

Furthermore, using the technique on page 168 of [10], we find that

$$-\nabla \iint_A \frac{\nabla'\cdot\boldsymbol{M}_1}{|\boldsymbol{r}-\boldsymbol{r}'|}\, ds' = \hat{\boldsymbol{u}}_x \frac{3\pi}{4\rho_o^3} = \hat{\boldsymbol{u}}_x \frac{\pi}{\alpha_{m1}}, \quad \text{in } A \tag{41}$$

where $\hat{\boldsymbol{u}}_x$ is a unit vector in the x direction and

$$\alpha_{m1} = \tfrac{4}{3}\rho_o^3 \tag{42}$$

is an eigenvalue of the magnetic polarizability tensor of the aperture. Hence, (29) is readily satisfied by $\boldsymbol{M}_1$ of (31) and (32) with eigenvalue

$$\lambda_1 = -\frac{3\pi(\eta_a k_a + \eta_b k_b)}{2k_a k_b\left(\eta_a k_b^2 + \eta_b k_a^2\right)\alpha_{m1}}. \tag{43}$$

Similarly, we have

$$\iint_A \boldsymbol{M}_2\, ds' = \hat{\boldsymbol{u}}_y, \quad \text{in } A \tag{44}$$

and

$$-\nabla \iint_A \frac{\nabla'\cdot\boldsymbol{M}_2}{|\boldsymbol{r}-\boldsymbol{r}'|}\, ds' = \hat{\boldsymbol{u}}_y \frac{3\pi}{4\rho_o^3} = \hat{\boldsymbol{u}}_y \frac{\pi}{\alpha_{m2}}, \quad \text{in } A \tag{45}$$

where $\hat{\boldsymbol{u}}_y$ is a unit vector in the y direction and

$$\alpha_{m2} = \alpha_{m1}. \tag{46}$$

Hence, (29) is also satisfied by $\boldsymbol{M}_2$ of (33) and (34) with eigenvalue

$$\lambda_2 = \lambda_1. \tag{47}$$

Finally, if the media in both regions are the same, say $\epsilon_b = \epsilon_a$, $\mu_b = \mu_a$, then λ_1 and λ_2 become

$$\lambda_2 = \lambda_1 = -\frac{9\pi}{8k_a^3\rho_o^3}. \tag{48}$$

Next, we consider $\boldsymbol{M}_3$ given by (35) and (36) and its corresponding eigenvalue equation (30). It can be readily established analytically that

$$\iint_A \boldsymbol{M}_3 |\boldsymbol{r}-\boldsymbol{r}'|^2\, ds' = -2\rho\hat{\boldsymbol{u}}_\varphi, \quad \text{in } A \tag{49}$$

and, using the technique on p. 168 of [10], that

$$\iint_A \frac{\boldsymbol{M}_3}{|\boldsymbol{r}-\boldsymbol{r}'|}\, ds' = \frac{3\pi}{4\rho_o^3}\rho\hat{\boldsymbol{u}}_\varphi = \frac{\pi}{2\alpha_e}\rho\hat{\boldsymbol{u}}_\varphi, \quad \text{in } A. \tag{50}$$

Here, $\hat{\boldsymbol{u}}_\varphi$ is a unit vector in the φ direction and

$$\alpha_e = \tfrac{2}{3}\rho_o^3 \tag{51}$$

is the electric polarizability of the aperture. It then follows that $\boldsymbol{M}_3$ given by (35) and (36) satisfies (30) with eigenvalue

$$\lambda_3 = \frac{3\pi(k_a\eta_b + k_b\eta_a)}{2\left(k_a^4\eta_b + k_b^4\eta_a\right)\alpha_e}. \tag{52}$$

Finally, if the media in both regions are the same, say $\epsilon_b = \epsilon_a$, $\mu_b = \mu_a$, then (52) reduces to

$$\lambda_3 = \frac{9\pi}{4k_a^3\rho_o^3}. \tag{53}$$

Note that in the case of an aperture with two identical media, $\lambda_3 = -2\lambda_1$.

V. Conclusions

A recently formulated eigenvalue equation for aperture problems has been specialized to the range of frequencies for which the maximum linear dimension of the aperture is much smaller than the wavelength. The result is a set of two eigenvalue equations which, aided by the edge condition on the contour bounding the aperture, constitutes an alternative set of equations for a derivation of the quasi-static distributions characterizing the aperture. Finally, the circular aperture case has been chosen to exemplify that the quasi-static solutions are indeed solutions of these low-frequency eigenvalue equations.

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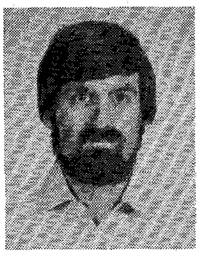

Yehuda Leviatan (S'81–M'82) was born in Jerusalem, Israel, on September 19, 1951. He received the B.Sc. and M.Sc. degrees in electrical engineering from the Technion–Israel Institute of Technology, Haifa, Israel, in 1977 and 1979, respectively, and the Ph.D. degree in electrical engineering from Syracuse University, Syracuse, NY, in 1982.

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